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A Discrete Characterization Theorem for the Discrete L_1 Linear Approximation Problem

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Let $C(X)$ be the set of real valued functions that are defined on a finite set of points X, and let A be an n-dimensional linear subspace of $C(X)$. For any f in $C(X)$, the element ϕ^* in A is a best L_1 approximation from A to f if it minimizes the expression

$$\sum_{x \in X} |f(x) - \phi(x)|, \qquad \phi \in A. \tag{1}$$

Two applications in which the calculation of ϕ^* is important are fitting to numerical data, and discrete models of continuous L_1 approximation problems. The calculation of a best L_1 approximation in the discrete case can be solved as a linear programming problem [1]. A solution always exists, but it need not be unique [2].

The purpose of this note is to express the conditions for the solution of the linear programming problem in terms of the original data. Thus a characterization theorem for ϕ^* is obtained, that is more useful than the usual characterization theorem [2], which is as follows.

THEOREM 1. *Let ϕ^* be a trial approximation from A to f, let $Z \subset X$ be the set of zeros*

$$Z = \{x : \phi^*(x) = f(x)\}, \tag{2}$$

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let Y contain the points of X that are not in Z, and let s be the sign function

$$s(x) = \begin{cases} 1, & f(x) > \phi^*(x) \\ -1, & f(x) < \phi^*(x) \end{cases}, \qquad x \in Y. \tag{3}$$

A necessary and sufficient condition for ϕ^* to be a best L_1 approximation from A to f is that the inequality

$$\left| \sum_{x \in Y} s(x)\, \phi(x) \right| \leqslant \sum_{x \in Z} |\phi(x)| \tag{4}$$

holds for all functions ϕ in A.

In practice this theorem may be of little help in determining whether a trial approximation is best, because the number of different functions ϕ that can occur in expression (4) is infinite. However, only a finite number of inequalities have to be tested to find out whether a linear programming problem is solved. The new characterization theorem that is presented is derived from this remark.

We let B be the linear subspace of A, whose functions take the value zero at all points of Z. If ψ is an element of B that satisfies the condition

$$\sum_{x \in Y} s(x)\, \psi(x) \neq 0, \tag{5}$$

then it follows, from the proof of Theorem 1 in [2], that the trial approximation ϕ^* can be improved by the addition of a multiple of ψ. To discover whether such improvements can be obtained, it is only necessary to check condition (5) for a set of functions $\{\psi\}$ that is a basis of B. We suppose that these tests fail to resolve whether ϕ^* is a best L_1 approximation from A to f. Then, because the addition of an element of B to ϕ in inequality (4) makes no difference, we may restrict the functions ϕ in expression (4) to any linear subspace of A that is complementary to B. Because we may regard the complementary subspace as a set of approximating functions that takes the place of A, we assume without loss of generality that the complementary subspace is A itself. This assumption gives the helpful property that, if ϕ is an element of A such that the numbers $\{\phi(x), x \in Z\}$ are all zero, then ϕ is the zero element. Hence the number of points in Z is at least the dimension of A, namely, n.

The characterization theorem that comes from linear programming is particularly elegant in the frequently occurring case when the number of points in Z is exactly n. It is as follows.

THEOREM 2. *Let the conditions of Theorem 1 be satisfied, let Z contain* *exactly n points $\{z_j;\ j = 1, 2,\dots, n\}$, and let the cardinal functions $\{l_i;\ i = 1, 2,\dots, n\}$ in A be defined by the equations*

$$l_i(z_j) = \delta_{ij}, \qquad i, j = 1, 2,\dots, n. \tag{6}$$

A necessary and sufficient condition for ϕ^* to be a best L_1 approximation from A to f is that the inequalities

$$\left| \sum_{x \in Y} s(x)\, l_i(x) \right| \leqslant 1, \qquad i = 1, 2,\dots, n, \tag{7}$$

are satisfied.

Proof. If ϕ^* is a best approximation, then, by Theorem 1, inequality (4) must hold when $\phi = l_i$. Hence condition (7) is necessary. To prove the converse result, we write a general element $\phi \in A$ in the form

$$\phi = \sum_{j=1}^{n} \lambda_j l_j. \tag{8}$$

Expressions (6), (7) and (8) imply the inequality

$$\begin{aligned} \left| \sum_{x \in Y} s(x)\, \phi(x) \right| &\leqslant \sum_{j=1}^{n} \left\{ |\lambda_j| \left| \sum_{x \in Y} s(x)\, l_j(x) \right| \right\} \\ &\leqslant \sum_{j=1}^{n} |\lambda_j| \\ &= \sum_{x \in Z} |\phi(x)|. \end{aligned} \tag{9}$$

It follows from Theorem 1 that condition (7) is sufficient for ϕ^* to be a best approximation.

The theorem is useful because it shows that, when its conditions are satisfied, then one can find out whether a trial approximation is best by testing only n inequalities. The equivalent statement in linear programming terms is that, if one requires the least value of a linear function that is defined on a convex polyhedron in the space of the variables, and if a trial vector of variables is at a vertex of the polyhedron, then this vector gives the solution if and only if the objective function cannot be reduced by a move along one of the edges of the polyhedron that pass through the vertex. Edges correspond to cardinal functions.

This point of view may be used to extend Theorem 2 to the case when the set Z contains more than n points. In geometrical terms we consider the situation where more than n faces of the polyhedron meet at the trial vertex.

Hence the number of edges that join at the vertex is greater than before. It is still true that one can discover whether the trial vertex is optimal by testing the change in the objective function just along these edges. Therefore a discrete characterization condition can be obtained. We now define a function l in A to be a cardinal function if there exists a subset of Z, Z_l say, that contains exactly $(n-1)$ points, and that is such that the equations

$$\begin{gathered} l(x)=0, \qquad x \in Z_l, \\ \sum_{x \in Z} |l(x)| = 1 \end{gathered} \tag{10}$$

determine l uniquely except for its overall sign. The trial approximation ϕ^* is optimal if and only if the inequality

$$\left| \sum_{x \in Y} s(x)\, l(x) \right| \leqslant 1 \tag{11}$$

is satisfied for all cardinal functions.

A disadvantage of these conditions is that the work of testing them can be very great when the number of points in Z is much larger than n. Therefore another characterization theorem is given. We let the points of Z be $\{z_j; j=1,2,\ldots,m\}$, $m>n$, and we suppose that they are ordered in any way that allows functions $\{l_i; i=1,2,\ldots,n\}$ in A to be defined by Eq. (6). The theorem depends on the remark that all the information that is needed to test whether ϕ^* is optimal is contained in the numbers $\{l_i(z_j);\ i=1,2,\ldots,n;\ j=n+1,n+2,\ldots,m\}$ and in the left hand sides of the inequalities (7).

THEOREM 3. *Let the conditions of Theorem 2 be satisfied, except that Z is the set $\{z_j; j=1,2,\ldots,m\}$, where $m>n$. A necessary and sufficient condition for ϕ^* to be a best L_1 approximation from A to f is that there exist real numbers $\{\theta_j; j=n+1,n+2,\ldots,m\}$, such that the inequalities*

$$|\theta_j| \leqslant 1, \qquad j=n+1,n+2,\ldots,m \tag{12}$$

and

$$\left| \sum_{x \in Y} s(x)\, l_i(x) + \sum_{j=n+1}^{m} \theta_j l_i(z_j) \right| \leqslant 1, \qquad i=1,2,\ldots,n, \tag{13}$$

are obtained.

Proof. We express a general element of A in the form (8). If inequalities (12) and (13) hold, then the left hand side of expression (4) satisfies the bound

$$\begin{aligned}\left|\sum_{x\in Y} s(x)\,\phi(x)\right| &\leqslant \left|\sum_{x\in Y} s(x)\,\phi(x) + \sum_{j=n+1}^{m} \theta_j \phi(z_j)\right| + \sum_{j=n+1}^{m} |\phi(z_j)| \\ &\leqslant \sum_{i=1}^{n} |\lambda_i| + \sum_{j=n+1}^{m} |\phi(z_j)| \\ &= \sum_{x\in Z} |\phi(x)|. \end{aligned} \tag{14}$$

It follows from Theorem 1 that conditions (12) and (13) are sufficient for ϕ^* to be a best approximation.

In order to prove that they are also necessary, we let ϕ^* be a best approximation, in which case Theorem 1 implies that the components of every vector $\boldsymbol{\lambda}$ in R^n satisfy the conditions

$$\left|\sum_{x\in Y} s(x) \sum_{i=1}^{n} \lambda_i l_i(x)\right| \leqslant \sum_{j=1}^{m} \left|\sum_{i=1}^{n} \lambda_i l_i(z_j)\right|. \tag{15}$$

The terms that depend on $\{z_j; j = n+1, n+2,..., m\}$ have to be transferred from the right to the left hand side of this inequality. The method that is used depends on the following lemma, which is proved later.

LEMMA. *Let ψ be a real, continuous convex function, defined on R^n, that satisfies the homogeneity condition*

$$\psi(\alpha\boldsymbol{\lambda}) = |\alpha|\,\psi(\boldsymbol{\lambda}), \qquad \boldsymbol{\lambda} \in R^n, \quad \alpha \in R. \tag{16}$$

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ be n-component vectors, such that the inequality

$$|\boldsymbol{\sigma}^T\boldsymbol{\lambda}| \leqslant \psi(\boldsymbol{\lambda}) + |\boldsymbol{\rho}^T\boldsymbol{\lambda}| \tag{17}$$

holds for all $\boldsymbol{\lambda}$ in R^n. Then the condition

$$|(\boldsymbol{\sigma} + \theta\boldsymbol{\rho})^T \boldsymbol{\lambda}| \leqslant \psi(\boldsymbol{\lambda}), \qquad \boldsymbol{\lambda} \in R^n, \tag{18}$$

is obtained, where θ is a constant in the interval $-1 \leqslant \theta \leqslant 1$.

By letting $\boldsymbol{\sigma}$ be the vector whose components have the values $\{\sum s(x)\, l_i(x); i = 1, 2,..., n\}$, by letting ψ be the function

$$\psi(\boldsymbol{\lambda}) = \sum_{j=1}^{m-1} \left|\sum_{i=1}^{n} \lambda_i l_i(z_j)\right|, \qquad \boldsymbol{\lambda} \in R^n, \tag{19}$$

and by letting the components of $\boldsymbol{\rho}$ have the values $\{l_i(z_m); i = 1, 2,..., n\}$, it follows by applying the lemma to expression (15) that the inequality

$$\left| \sum_{i=1}^{n} \lambda_i \left\{ \sum_{x \in Y} s(x)\, l_i(x) + \theta_m l_i(z_m) \right\} \right| \leqslant \sum_{j=1}^{m-1} \left| \sum_{i=1}^{n} \lambda_i l_i(z_j) \right| \tag{20}$$

holds, where θ_m satisfies condition (12). This method is used inductively to deduce the bound

$$\left| \sum_{i=1}^{n} \lambda_i \left\{ \sum_{x \in Y} s(x)\, l_i(x) + \sum_{j=n+1}^{m} \theta_j l_i(z_j) \right\} \right| \leqslant \sum_{i=1}^{n} |\lambda_i|, \tag{21}$$

where the moduli of the numbers $\{\theta_j; j = n+1, n+2,..., m\}$ do not exceed one. It is now straightforward to obtain condition (13) from the special case when $\boldsymbol{\lambda}$ is a coordinate vector.

Proof of Lemma. By Corollary 13.2.1 of [3], there exists a closed convex set C in R^n, such that ψ is the function

$$\psi(\boldsymbol{\lambda}) = \max\{\boldsymbol{\mu}^T\boldsymbol{\lambda}; \boldsymbol{\mu} \in C\}, \qquad \boldsymbol{\lambda} \in R^n. \tag{22}$$

Therefore expression (17) implies the bound

$$\boldsymbol{\sigma}^T\boldsymbol{\lambda} \leqslant \max\{\boldsymbol{\mu}^T\boldsymbol{\lambda}; \boldsymbol{\mu} \in C\} + |\boldsymbol{\rho}^T\boldsymbol{\lambda}|. \tag{23}$$

Hence the inequality

$$\boldsymbol{\sigma}^T\boldsymbol{\lambda} \leqslant \max\{\boldsymbol{\tau}^T\boldsymbol{\lambda}; \boldsymbol{\tau} \in C^+\}, \qquad \boldsymbol{\lambda} \in R^n, \tag{24}$$

holds, where C^+ is the convex set

$$C^+ = \{\boldsymbol{\tau}: \boldsymbol{\tau} = \boldsymbol{\mu} + \alpha\boldsymbol{\rho}, \boldsymbol{\mu} \in C, -1 \leqslant \alpha \leqslant 1\}. \tag{25}$$

Expression (24) and Theorem 13.1 of [3] show that $\boldsymbol{\sigma}$ is in the set C^+. Therefore there exists θ in $[-1, 1]$ such that the vector $(\boldsymbol{\sigma} + \theta\boldsymbol{\rho})$ is in C. It follows from equation (22) that the condition

$$(\boldsymbol{\sigma} + \theta\boldsymbol{\rho})^T \boldsymbol{\lambda} \leqslant \psi(\boldsymbol{\lambda}), \qquad \boldsymbol{\lambda} \in R^n, \tag{26}$$

is obtained. The required inequality (18) is now a consequence of the fact that $\psi(\boldsymbol{\lambda})$ and $\psi(-\boldsymbol{\lambda})$ are equal.

Theorem 3 is useful in practice, because the tests (12) and (13) for a best discrete L_1 approximation can be made conveniently by a linear programming calculation in only $(m-n)$ variables. It is straightforward to generalize the theorems of this paper to the case when one requires an element in A that minimizes the expression

$$\sum_{x \in X} \omega(x)\,|f(x) - \phi(x)|, \qquad \phi \in A, \tag{27}$$

where the numbers $\{\omega(x); x \in X\}$ are given positive weights.

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